ON THE FUNCTIONAL EQUATION FOR THE SINE. ADDITIONAL NOTE.

By Edward B. Van Vleck.

In vol. 10 of the Annals, p. 161, I brought forward the following functional equation for the sine:

$$f(x - y + A) - f(x + y + A) = 2f(x)f(y). \tag{I}$$

In the second of the two proofs there given for the definition of the sine by means of this equation, one step should have been supplied to the reader between 7) and 8) to justify in the second equation of 8) the exclusion of the values $f(x) = \pm 1$ for $0 < x < B < A$. The justification of this exclusion runs as follows.

Suppose k to be a value between 0 and A for which $f(k) = \pm 1$. Then by my equation * (2),

$$f(k + A) = 0.$$

Putting $y = k$ in (1) we have

$$f(x - k + A) + f(x + k + A) = 0.$$

But for $y = k$ equation (I) becomes

$$f(x - k + A) - f(x + k + A) = \pm 2f(x).$$

The combination of these two equations gives

$$f(x + k + A) = \mp f(x).$$

This is equivalent to

$$f(x + k - A) = \pm f(x),$$

since the addition of $2A$ to the argument of the function changes its sign. Hence if x is replaced in (I) by $x + k - A$, there results the equation

$$f(x - y + k) - f(x + y + k) = \pm 2f(x)f(y),$$

where the upper or lower sign holds in the right hand member according as $f(x) = +1$ or $f(x) = -1$. In the former case the equation is the same as (I) with k in place of A, while in the latter case we have merely to put $f_1(x) \equiv -f(x)$ to make it the same. We may therefore suppose A in (I) to be the smallest value of $k > 0$ for which $f(k) = \pm 1$. In other words, between 0 and A we have

$$-1 < f(x) < 1.$$

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* The equations referred to are

$$f(x - y + A) + f(x + y + A) = 2f(x + A)f(y + A), \tag{1}$$

$$1 = f^2(x) + f^2(x + A). \tag{2}$$